Solutions of mid-sem exam, probability theory I, B. math. I year, 2008-09, ISI-BC

Solution 1:

$$
\begin{aligned}
P\{A \subset B\} &= \sum_{k=0}^{n} P(A \subset B : |B| = k) \cdot P(|B| = k) \\
&= \sum_{k=0}^{n} \frac{1}{2^n} \binom{n}{k} \sum_{r=0}^{k} \frac{1}{2^n} \binom{k}{r} \\
&= \frac{1}{4^n} \sum_{k=0}^{n} \binom{n}{k} 2^k \\
&= \frac{3^n}{4^n}.
\end{aligned}
$$

Solution 2:
Probability that there is no 1 in n trials is, $(p_0 + p_2)^n = (1 - p_1)^n$.
Probability that there is no 2 in n trials is, $(p_0 + p_1)^n = (1 - p_2)^n$.
Therefore, probability that outcomes 1 and 2 both occur atleast once is,
$1 - (1 - p_1)^n - (1 - p_2)^n + p_0^n$. Because we were excluding the case 2 times where only 0 appears in n trials.

Solution 3:
We will prove it by mathematical induction. Assume it holds for $(n - 1)$ trials. In the n th trial, it could be-
(i) Either, we have odd number of success, we will need the last trial to be a success to have even number of success.
(ii) OR, we have even number of success, then we need the last trial to be failure.

$$
\begin{aligned}
P_n &= (1 - P_{n-1})p + P_{n-1}(1 - p) \\
&= \left(1 - \frac{1 + (1 - 2p)^{n-1}}{2}\right) p + \frac{1 + (1 - 2p)^{n-1}}{2}(1 - p) \\
&= p - p\frac{1 + (1 - 2p)^{n-1}}{2} + \frac{1 + (1 - 2p)^{n-1}}{2} - p\frac{1 + (1 - 2p)^{n-1}}{2} \\
&= \frac{1}{2} - p(1 - 2p)^{n-1} + \frac{1}{2}(1 - 2p)^{n-1} \\
&= \frac{1}{2} + (1 - 2p)^{n-1}(\frac{1}{2} - p) \\
&= \frac{1 + (1 - 2p)^n}{2}.
\end{aligned}
$$

Solution 4:

$$\begin{aligned}
E[\frac{1}{x+1}] &= \sum_{x=0}^{n} \frac{1}{x+1}\binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \frac{1}{x+1}\frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \frac{1}{(n+1)p}\frac{(n+1)!}{(x+1)!(n-x)!} p^{x+1} (1-p)^{n-x} \\
&= \frac{1}{(n+1)p}\sum_{x=0}^{n} \frac{(n+1)!}{(x+1)!(n-x)!} p^{x+1} (1-p)^{n-x} \\
&= \frac{1}{mp}\sum_{y=1}^{m} \frac{m!}{y!(m-y)!} p^{y} (1-p)^{m-y} \qquad [\text{assuming } x+1=y \text{ and } n+1=m] \\
&= \frac{1}{mp}\sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^{y} (1-p)^{m-y} - \frac{1}{mp}(1-p)^m \\
&= \frac{(p+(1-p))^m}{mp} - \frac{(1-p)^m}{mp} \\
&= \frac{1-(1-p)^m}{mp} \\
&= \frac{1-(1-p)^{n+1}}{(n+1)p}. \qquad [\text{by replacing } m=n+1]
\end{aligned}$$

Solution 5:

$$\begin{aligned}
&\lambda E[(X+1)^{n-1}] \\
&= \lambda \sum_{x=0}^{\infty} (x+1)^{n-1} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} (x+1)^{n} \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} \\
&= \sum_{x+1=1}^{\infty} (x+1)^{n} \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} \\
&= \sum_{x=0}^{\infty} x^{n} \frac{e^{-\lambda}\lambda^{x}}{x!} \\
&= E[X^n].
\end{aligned}$$

Now, if X is a Poisson random variable with parameter λ, then $E[X] = \lambda$. And, by the above formula,

$$\begin{aligned} E[X^2] &= \lambda E[(X+1)] \\ &= \lambda E[X] + \lambda \\ &= \lambda^2 + \lambda. \end{aligned}$$

Similarly, by applying the above formula,

$$\begin{aligned} E[X^3] &= \lambda E[(X+1)^2] \\ &= \lambda E[X^2 + 2X + 1] \\ &= \lambda E[X^2] + 2\lambda E[X] + \lambda \\ &= \lambda^3 + \lambda^2 + 2\lambda^2 + \lambda \\ &= \lambda^3 + 3\lambda^2 + \lambda. \end{aligned}$$

Solution 6:

Let there are i $(1 \le i \le n)$ elements in the choosen subset. So,

$$P(X = i) = \frac{\binom{n}{i}}{2^n - 1}.$$

Therefore,

$$\begin{aligned} E[X] &= \sum_{i=0}^{n} i \frac{\binom{n}{i}}{2^n - 1} \\ &= \frac{n}{2^n - 1} \sum_{i=1}^{n} \binom{n-1}{i-1} \\ &= \frac{n}{2^n - 1} 2^{n-1} \\ &= \frac{n}{2 - \left(\frac{1}{2}\right)^{n-1}}. \end{aligned}$$

$$\begin{aligned} Var(X) &= E[X^2] - E^2[X] \\ &= E[X(X-1) + X] - E^2[X] \\ &= E[X(X-1)] + E[X] - E^2[X]. \end{aligned}$$

Now,

$$\begin{aligned} E[X(X-1)] &= \sum_{i=0}^{n} i(i-1) \frac{\binom{n}{i}}{2^n - 1} \\ &= \frac{n(n-1)}{2^n - 1} \sum_{i=2}^{n} \binom{n-2}{i-2} \\ &= \frac{n(n-1)}{2^n - 1} 2^{n-2}. \end{aligned}$$

Therefore,

$$\begin{aligned}
Var(X) &= E[X(X-1)] + E[X] - E^2[X] \\
&= \frac{n(n-1)}{2^n-1}2^{n-2} + \frac{n}{2^n-1}2^{n-1} - \frac{n^2}{(2^n-1)^2}2^{2n-2} \\
&= \frac{(n^2-n)2^{n-2}(2^n-1) + n2^{n-1}(2^n-1) - n^2 2^{2n-2}}{(2^n-1)^2} \\
&= \frac{(n^2-n)(2^{2n-2}-2^{n-2}) + n2^{2n-1} - n2^{n-1} - n^2 2^{2n-2}}{(2^n-1)^2} \\
&= \frac{n^2 2^{2n-2} - n2^{2n-2} - n^2 2^{n-2} + n2^{n-2} + n2^{2n-1} - n2^{n-1} - n^2 2^{2n-2}}{(2^n-1)^2} \\
&= \frac{n2^{2n-2} - n^2 2^{n-2} - n2^{n-2}}{(2^n-1)^2} \quad [\text{since, } n2^{n-1} = 2n2^{n-2} \text{ and } n2^{2n-1} = 2n2^{2n-2}] \\
&= \frac{n2^{2n-2} - n(n+1)2^{n-2}}{(2^n-1)^2}.
\end{aligned}$$

Solution 7:

(a) From the given conditions $P\{X = 2\} = \frac{1}{2} \cdot \frac{1}{3}$.
Similarly $P\{X = i\} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{i-1}{i} \cdot \frac{1}{i+1} = \frac{1}{i(i+1)}$.
Therefore $P\{X > i\} = \sum_{k=i+1}^{\infty} \frac{1}{k(k+1)}$, where $i \geq 1$.

(b)

$$\begin{aligned}
P\{X < \infty\}^c &= \lim_{i\to\infty} P(X \geq i) \\
&= \lim_{i\to\infty} \sum_{k=i}^{\infty} \frac{1}{k(k+1)} \\
&= \lim_{i\to\infty} \lim_{N\to\infty} \sum_{k=i}^{N} \left\{ \frac{1}{k} - \frac{1}{k+1} \right\} \\
&= \lim_{i\to\infty} \frac{1}{i} \\
&= 0.
\end{aligned}$$

Therefore $P\{X < \infty\} = 1$.

(c)

$$E[X] = \sum_{i=1}^{\infty} i \cdot \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \frac{1}{i+1} = \infty.$$